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Compressibility effects on homogeneous isotropic turbulence using Schur decomposition of the velocity gradient tensor

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THUWAL, KINGDOM OF SAUDI ARABIA

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OVERVIEW

1. MOTIVATION & PURPOSE
2. NUMERICAL METHODS & DNS DATABASE
3. NUMERICAL RESULTS
4. CONCLUSION & PERSPECTIVES

MOTIVATION & PURPOSE

Motivation

- Study of the velocity gradient tensor (VGT) $\mathcal{A}_{ij} = \partial \mathcal{U}_i / \partial x_j$ in turbulent flows.

  → VGT contains all the necessary information

  → its role may be made explicity by taking the spatial gradient of the Navier-Stokes equations

$$\partial \boldsymbol{\mathcal{A}}/\partial t + \boldsymbol{\mathcal{U}} \cdot \boldsymbol{\nabla} \mathcal{A} = -\boldsymbol{\mathcal{A}}^2 - \boldsymbol{\mathcal{H}} + \nu \nabla^2 \mathcal{A}, \text{ where } \mathcal{H}_{ij} = \partial^2 \mathcal{P} / \partial x_i \partial x_j$$

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Study of the velocity gradient tensor

Hermitian and skew-Hermitian decomposition approach

$$\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{S}}^{\mathcal{A}} + \boldsymbol{\Omega}^{\mathcal{A}}, \; \boldsymbol{\omega} = \nabla \times \boldsymbol{\mathcal{U}}$$

alignment between the vorticity vector/scalar gradient and the eigenvectors of the strain rate tensor $\boldsymbol{\mathcal{S}}^{\mathcal{A}}$

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Eigenvalue-based approach[¶]

$$\lambda_i^3 + P^{\mathcal{A}} \lambda_i^2 + Q^{\mathcal{A}} \lambda_i + R^{\mathcal{A}} = 0$$

$P^{\mathcal{A}} Q^{\mathcal{A}} R^{\mathcal{A}}$ invariants with the identites

$$P^{\mathcal{A}} = \sum_i \lambda_i, \; Q^{\mathcal{A}} = \sum_{i<j} \lambda_i \lambda_j, \; R^{\mathcal{A}} = \prod_i \lambda_i$$

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Purpose

- Unify eigenvalue-based approach and Hermitian/Skew-Hermitian approaches through a recently proposed Schur decomposition approach*

$$\mathcal{A} = \mathcal{B}^{\mathcal{A}} + \mathcal{C}^{\mathcal{A}}$$

- Use this decomposition to evaluate the influence of the compressibility on some statistical properties of the turbulent structures.

Conceptual Overview

- Schur decomposition $\mathcal{A} = \mathcal{U}\mathcal{T}\mathcal{U}^*$ which $\mathcal{T} = \Lambda + \mathcal{N}$
  - $\rightarrow \Lambda$ is diagonal matrix whose elements correspond to the eigenvalues of $\mathcal{A}$
  - $\rightarrow \mathcal{N}$ is an upper triangular matrix that represents the non-normal part of $\mathcal{A}$
- Tensors of the additive decomposition $\mathcal{A} = \mathcal{B}^{\mathcal{A}} + \mathcal{C}^{\mathcal{A}}$ are defined as

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NUMERICAL METHODS & DNS DATABASE

Numerical solver

- Cartesian Navier-Stokes solver, three-dimensional, compressible, unsteady, viscous solver
  → Convective fluxes are discretized using a 7^{th} accurate hybrid upwinded-WENO scheme[†] .
  → Molecular fluxes are discretized using 8^{th} order accurate centred difference scheme.
- Temporal integration is performed using 3^{rd} accurate total variation diminishing RK scheme.

DNS database

Resolution	Re_λ	Ma_t	$\mathcal{U}'$	$\langle \varepsilon/\rho \rangle$	$\Delta x/\eta$	$\mathcal{L}_t/\eta$	λ/η	$\mathcal{S}k_3$	$\mathcal{F}l_3$
512^3	100	0.12	0.54	0.11	1.18	151	19.80	-0.43	5.50
512^3	100	0.32	0.53	0.10	1.17	154	19.79	-0.45	5.64
512^3	100	0.50	0.53	0.10	1.15	154	19.59	-0.50	5.53
512^3	100	0.59	0.46	0.11	1.29	181	19.59	-0.51	5.94
512^3	100	0.73	0.45	0.09	1.35	175	19.37	-0.71	6.10
512^3	100	0.89	0.45	0.07	1.41	172	19.05	-1.18	8.81

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Estimate of the non-normality effects

Use of standardized difference to understand when the non-normal effects are significant

$$\kappa_{\mathcal{B},\mathcal{C}} = \frac{\|\mathcal{B}\| - \|\mathcal{C}\|}{\|\mathcal{B}\| + \|\mathcal{C}\|}.$$

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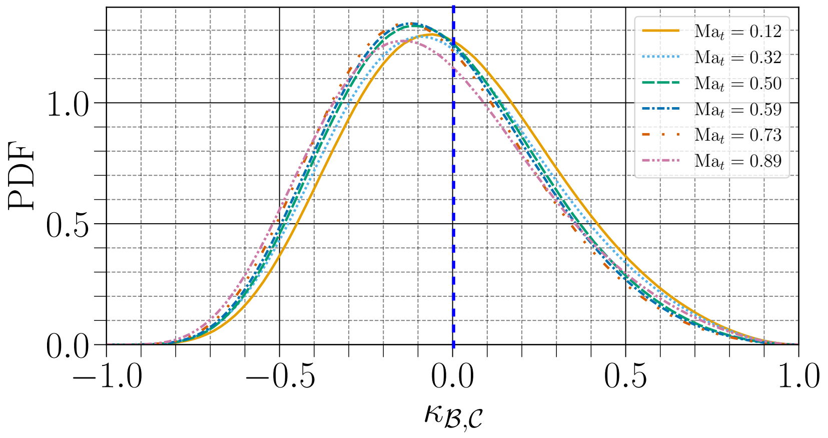

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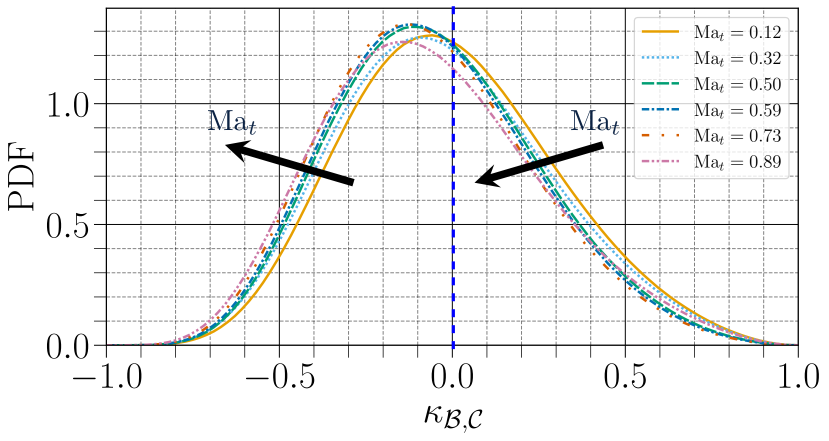
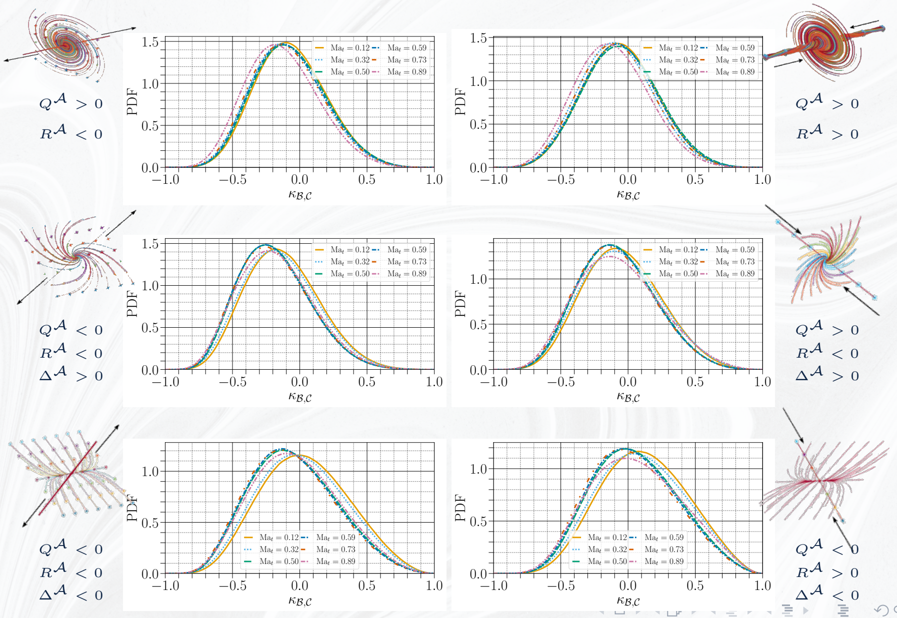
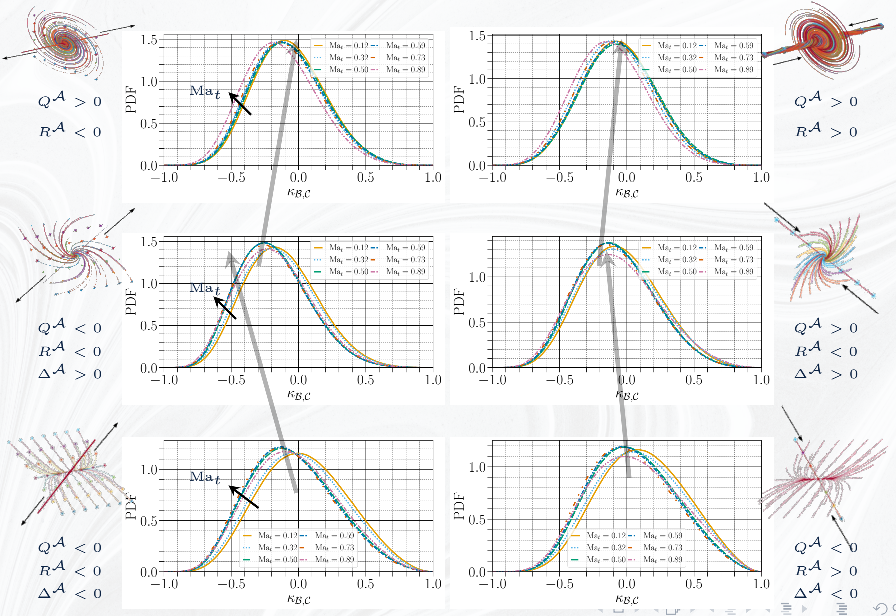


NUMERICAL RESULTS





$Q^{\mathcal{A}} > 0$
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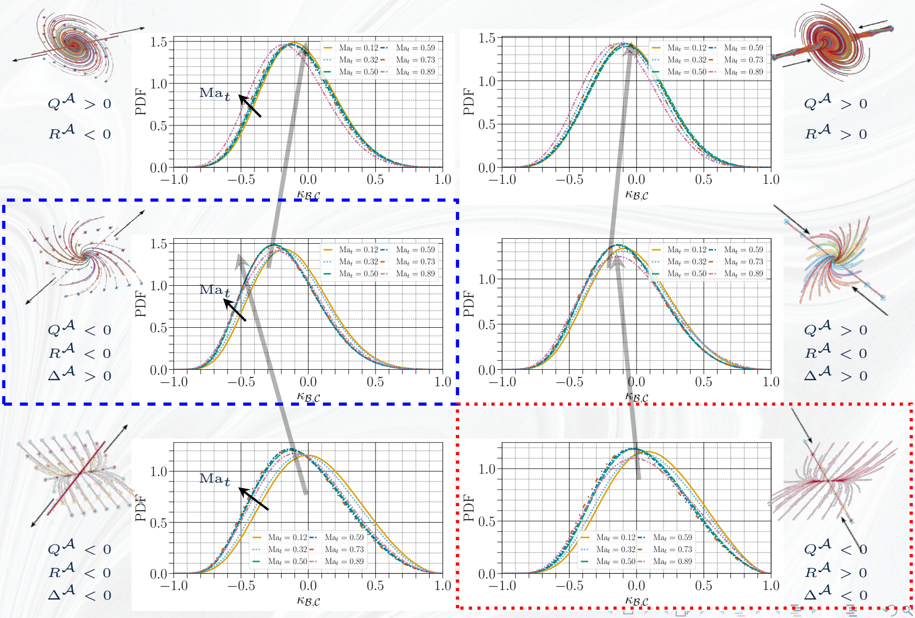

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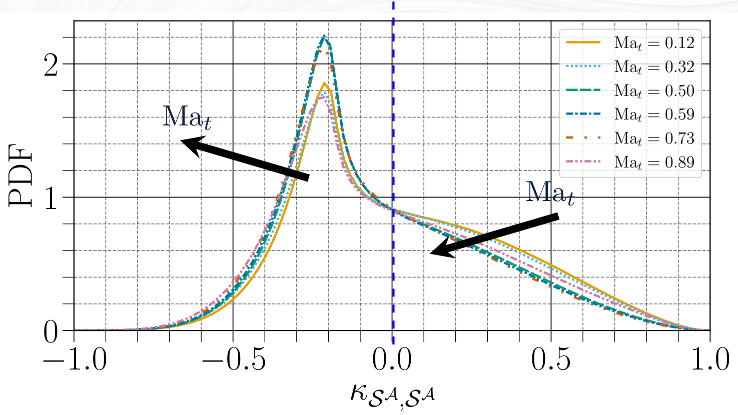

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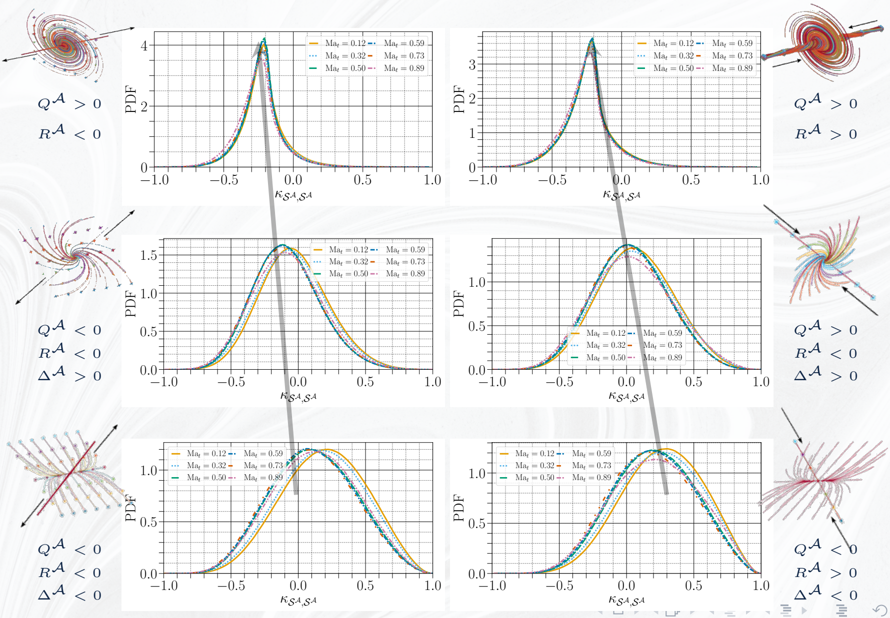

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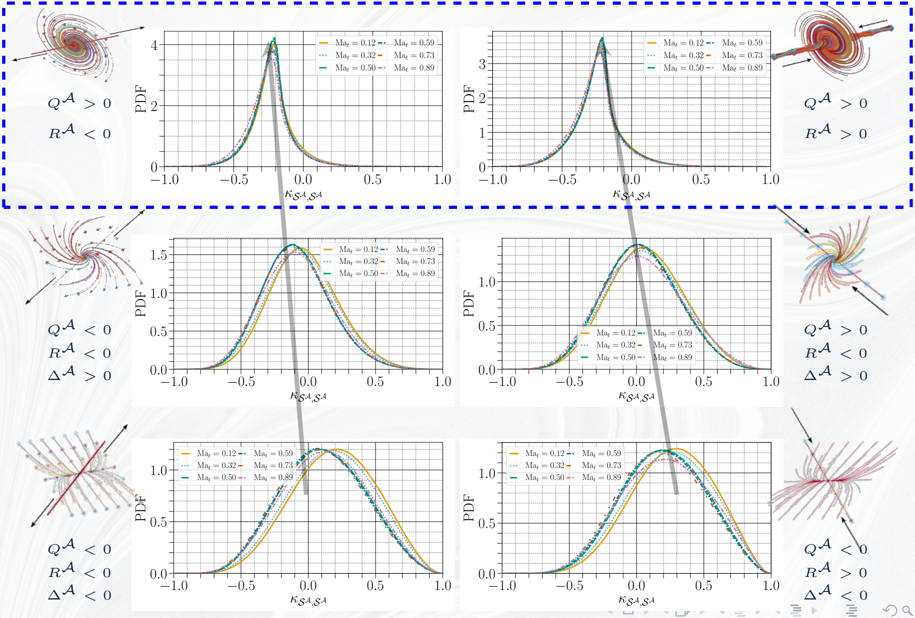

Intermediate eigenvalue parameter

Normalization of the intermediate eigenvalue parameter of Lund* $e_{\mathcal{A}} = \frac{3\sqrt{6}R^{\mathcal{S}^{\mathcal{A}}}}{\left(-2Q^{\mathcal{S}^{\mathcal{A}}}\right)^{3/2}}$

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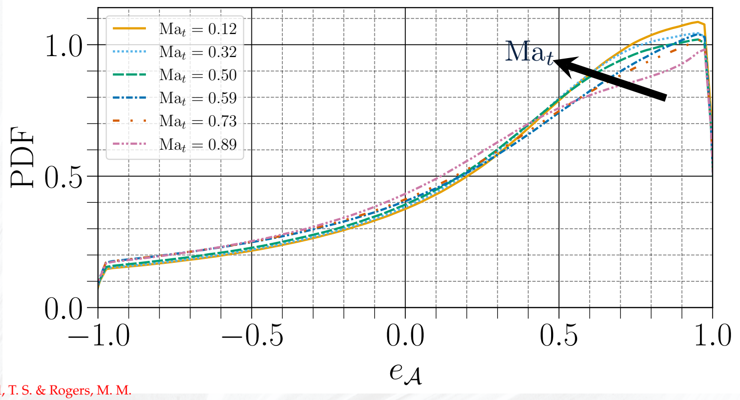

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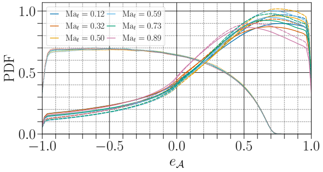

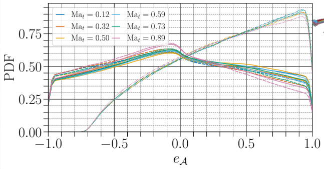

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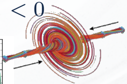

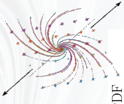

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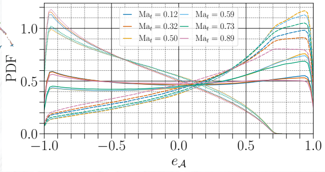

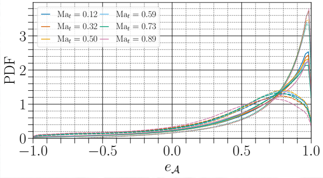

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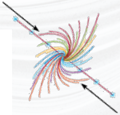

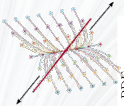

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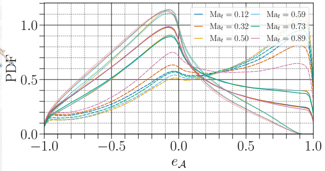

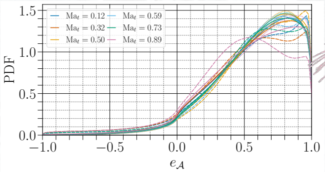

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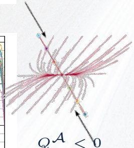

Numerical results

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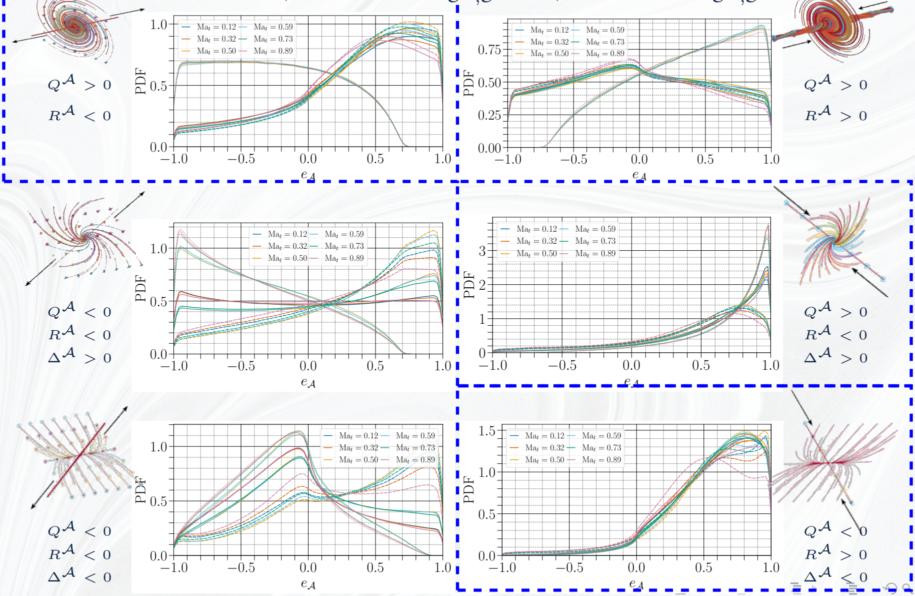

Conclusion

- This approach provides a means to link the eigenvalue and strain-rotation based approaches to studying the VGT.
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Conclusion & Perspectives

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Perspectives

- The next stage is to revisit some of the modeling approaches that already exist and to see if we can develop them in a more effective fashion using this approach.
- Moving beyond the HIT test case.

Thanks for your attention

Acknowledgments

جامعة الملك عبدالله
للعلوم والتقنية
King Abdullah University of
Science and Technology